AIAA JOURNAL
Vol. 36, No. 3, March 1998

Wavelet-Based Model for Stochastic Analysis of Beam Structures

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A wavelet-based model for stochastic analysis of beam structures is presented. In this model, the random processes representing the stochastic material and geometric properties are treated as stationary Gaussian processes with specified mean and correlation functions. Using the Karhunen–Loève expansion, the process is represented as a linear sum of orthonormal eigenfunctions with uncorrelated random coefficients. The correlation and the eigenfunctions are approximated as truncated linear sums of compactly supported orthogonal wavelets, and the integral eigenvalue problem is converted to a finite dimensional eigenvalue problem. The energy-principle-based finite element approach is used to obtain the equilibrium and boundary conditions. Neumann expansion of the stiffness matrix is used to write the nodal displacement vector in terms of random coefficients. The expectation operator is applied to the nodal displacements and their squares to obtain the mean and standard deviation of the displacements. Studies show that the results obtained using this method compare well with Monte Carlo and semianalytical techniques.

I. Introduction

A NEW, wavelet-based technique for stochastic finite element analysis (SFEA) of beam structures is presented. The concept of orthonormal bases of compactly supported wavelets was introduced first by Daubechies.[1] Since then, wavelets have been applied in many fields, which include image coding, edge detection in images, seismic signal analysis, acoustic signal detection, and vibrational diagnosis of machine parts. The number of applications of wavelets in various areas of mathematics, science, and engineering is growing every day because of the advantages that the wavelets offer over other techniques. In the field of numerical analysis, wavelets have been used to develop fast numerical algorithms for integral and differential operators,[2,3] fast multiplication of structured dense matrices to arbitrary vectors,[4,5] and to solve partial differential equations.[6,7] Beylkin[8] indicates that for the construction of the generalized inverse of certain matrices, the fast wavelet transform technique can be four orders of magnitude faster than the singular value decomposition technique used currently. Several investigators have used wavelets for multiresolution analysis of stochastic processes.[9–12]

This brief account of wavelets and their applications is by no means complete. A review of science citations and engineering indexes indicates that over 1000 papers on wavelets have been published in the past seven years, and the number of applications of wavelets is increasing very rapidly.

Contrary to that of wavelets, the field of probabilistic/stochastic analysis of structures and other engineering systems is relatively old. Research conducted in the past in this area was confined to systems with regular geometry. Recently, various finite and boundary element techniques have been developed that allow stochastic analysis of engineering systems with arbitrary shapes. Methods used in this field include the Monte Carlo simulation approach,[13,14] the perturbation method,[15,16] the Neumann expansion method,[17–19] and the Karhunen–Loève (K–L) expansion-based finite element method (FEM).[20,21] Ghanem and Spanos[22] have written a book on the spectral stochastic finite element approach for structural systems. This book also provides a comprehensive review of the literature on the subject published prior to 1991. Several investigators have used the boundary element method to solve stochastic engineering systems.[23–30]

The above references clearly indicate the importance of stochastic analysis of engineering systems. The importance of wavelets in SFEA lies in the fact that SFEA of engineering systems involves several matrix multiplications and solution of simultaneous equations, and transformation and inverse transformation of functions and equations from one base to another[22] (see also the formulation presented in Secs. II and III), all of which can be performed very efficiently using wavelet bases.[1,2] Thus, it is clear that application of wavelets in SFEA will significantly improve the computational efficiency of the numerical schemes used currently.

We show that this new wavelet-based technique for SFEA of beam structures requires the generation of only a few new matrices for stochastic analysis. Because of its simplicity, this technique can be implemented in an existing finite element code without major modification. The technique employs wavelet-based K–L expansion of stochastic processes to compute stochastic matrices and Neumann expansion of the stiffness matrix to write the displacement vector in terms of random coefficients. Although only beam structures are considered, the method can be applied without major modification to other engineering systems. For a semianalytical approach similar to the one presented here, the readers are referred to Ref. 22.

II. General Stochastic FEM

The total potential energy of a beam structure is given as

$$\pi = \frac{1}{2}\int_0^L EI(x)\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L p(x)y(x)\,\mathrm{d}x \qquad (1)$$

where $EI(x)$ and L are the bending stiffness and the length of the beam, respectively, and $p(x)$ and $y(x)$ are the transverse load and the transverse displacement functions, respectively. Observe that EI is a function of x, and thus the formulation allows the material and the geometric properties to vary along the length. In the past, $EI(x)$ was treated as a known function. In practice, the geometry and the material properties vary randomly from sample to sample and from point to point in a sample. As a result, $EI(x)$ is described by a stochastic process. For simplicity, it is assumed that

$$EI(x) = (EI)_o[1 + f(x)] \qquad (2)$$

Received April 25, 1997; revision received Oct. 10, 1997; accepted for publication Oct. 21, 1997.

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where $(EI)_o$ is the constant mean function and $f(x)$ is the specified random process that describes the stochastic behavior of the geometry and the material properties. The definition indicates that $f(x)$ has zero mean function. It is further assumed that the boundary conditions are deterministic and the function $f(x)$ is a stationary Gaussian process with a specified correlation function $R_f(x, u)$, where u is another point along the beam. This assumption allows one to compute the higher moment characteristics in a closed form. The assumptions do not limit the approach presented here. If the mean function of $EI(x)$ is a function of x, $(EI)_o$ can be replaced by the mean function. Random boundary conditions introduce only parametric randomness that can be treated as random initial conditions.[31] Furthermore, the approach presented can be extended to nonstationary and truncated white-noise processes.[22] These assumptions are made for simplicity.

The process $f(x)$ can be written in terms of K–L series as

$$f(x) = \lim_{N\to\infty} \sum_{i=1}^{N} \epsilon_i \phi_i(x) \tag{3}$$

where $\{\phi_i(x) \mid i = 1, \ldots, \infty\}$ is a complete set of orthonormal eigenfunctions and $\{\epsilon_i \mid i = 1, \ldots, \infty\}$ is a set of uncorrelated variables.[32] Equation (3) converges in a mean-square sense. The eigenfunctions satisfy the following integral eigenvalue problem:

$$\lambda_i \phi_i(x) = \int_0^L R_f(x, u)\phi_i(u)\, du \tag{4}$$

where λ_i $(i = 1, \ldots, \infty)$ are the eigenvalues associated with the eigenfunctions $\phi_i(x)$. Because the process $f(x)$ has zero mean function, it follows that the mean value of ϵ_i $(= E[\epsilon_i])$ is also zero. Here, E is the expectation operator. The coefficients ϵ_i $(i = 1, \ldots, \infty)$ satisfy the following identity[32]:

$$\lambda_i \delta_{ij} = E[\epsilon_i \epsilon_j] \tag{5}$$

where δ_{ij} is the Kronecker delta and $\lambda_i = E[\epsilon_i^2]$ is the expected value of the energy of the process $f(x)$ along the $\phi_i(x)$. Equation (5) gives, in some sense, the probability characteristic and relative importance of ϵ_i, and thus a guideline for approximating $f(x)$. It is implicitly assumed that λ_i decreases as i increases. One of the advantages of the K–L expansion is that it provides a second moment characterization of $f(x)$ in terms of uncorrelated random variables. Equations (3–5) and further discussion of the K–L expansion appear in Ref. 32.

For the stochastic FEM, $f(x)$ is approximated in terms of a truncated series as

$$f(x) = \sum_{i=1}^{N} \epsilon_i \phi_i(x) \tag{6}$$

where N is the number of terms selected in the series. Using Eqs. (1), (2), and (6), the approximate stochastic finite element equations for displacement u can be written as[22]

$$\left[K^{(0)} + \sum_{i=1}^{N} \epsilon_i K^{(i)}\right] \boldsymbol{u} = \boldsymbol{P} \tag{7}$$

where the deterministic stiffness matrix $K^{(0)}$ and the stochastic stiffness matrices $K^{(i)}$ are given as

$$K^{(0)} = \int_0^L B^T E I_{(0)} B\, dx \tag{8}$$

$$K^{(i)} = \int_0^L B^T E I_{(0)} \phi_i(x) B\, dx, \qquad i = 1, \ldots, N \tag{9}$$

Matrix $K^{(0)}$ is generally available in a finite element code. However, matrices $K^{(i)}$ $(i = 1, \ldots, N)$ are new, and they must be generated for stochastic analysis.

For simplicity, we assume homogeneous boundary conditions. The formulation, however, can be extended very easily for nonhomogeneous boundary conditions.[22] Using this assumption, Eq. (7) can be written as

$$\left[\bar{K}^{(0)} + \sum_{i=1}^{N} \epsilon_i \bar{K}^{(i)}\right] \bar{\boldsymbol{u}} = \bar{\boldsymbol{P}} \tag{10}$$

where the deterministic stiffness matrix $\bar{K}^{(0)}$, the ith stochastic stiffness matrix $\bar{K}^{(i)}$, the displacement vector $\bar{\boldsymbol{u}}$, and the load vector $\bar{\boldsymbol{P}}$ are obtained after imposing the boundary conditions in Eq. (7).

Equation (10), which governs the behavior of the discretized stochastic system, often is used in Monte Carlo simulations. This equation can be modified further by observing the fact that for structural systems and specifically for the beam structure considered here, matrix $\bar{K}^{(0)}$ is positive definite and, therefore, its inverse exists. Multiplying both sides of Eq. (10) by $[\bar{K}^{(0)}]^{-1}$, one obtains

$$\left[I + \sum_{i=1}^{N} \epsilon_i Q^{(i)}\right] \bar{\boldsymbol{u}} = \boldsymbol{b} \tag{11}$$

where

$$Q^{(i)} = \left[\bar{K}^{(0)}\right]^{-1} \bar{K}^{(i)} \tag{12}$$

and

$$\boldsymbol{b} = \left[\bar{K}^{(0)}\right]^{-1} \bar{\boldsymbol{P}} \tag{13}$$

The displacement vector $\bar{\boldsymbol{u}}$ now can be written as

$$\bar{\boldsymbol{u}} = \left[I + \sum_{i=1}^{N} \epsilon_i Q^{(i)}\right]^{-1} \boldsymbol{b}$$

A Neumann expansion of the preceding equation leads to

$$\bar{\boldsymbol{u}} = \sum_{j=1}^{\infty} (-1)^j \left[\sum_{i=1}^{N} \epsilon_i Q^{(i)}\right]^j \boldsymbol{b} \tag{14}$$

The series expansion in Eq. (14) exists, provided that

$$\left\| \left[\sum_{i=1}^{N} \epsilon_i Q^{(i)}\right] \right\| < 1 \tag{15}$$

where $\|\cdot\|$ denotes some matrix norm. Theoretically, ϵ_i can take any value, causing Eq. (15) to fail. For example, a large negative value of an ϵ_i may lead to a negative value for the stiffness of the beam. However, this is not the case in actual situations. In practical cases the values of ϵ_i are found in a given range for which Eq. (15) is satisfied.

Using Eq. (14), $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^T$ can be written in expanded form as

$$\bar{\boldsymbol{u}} = \left[I - \sum_{i=1}^{N} \epsilon_i Q^{(i)} + \sum_{i=1}^{N}\sum_{j=1}^{N} \epsilon_i \epsilon_j Q^{(i)} Q^{(j)} - \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \epsilon_i \epsilon_j \epsilon_k Q^{(i)} Q^{(j)} Q^{(k)} + \cdots\right] \boldsymbol{b} \tag{16a}$$

and

$$\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^T = \left[B - \sum_{i=1}^{N} \epsilon_i \left(Q^{(i)} B + B Q^{(i)T}\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} \epsilon_i \epsilon_j \left(Q^{(i)} Q^{(j)} B + Q^{(i)} B Q^{(j)T} + B Q^{(i)T} Q^{(j)T}\right) - \cdots\right] \tag{16b}$$

where $B = \boldsymbol{b}\boldsymbol{b}^T$. Applying the expectation operator to Eqs. (16a) and (16b) and using the probabilistic characteristics of ϵ_i, we obtain

$$E[\bar{\boldsymbol{u}}] = \left[I + \sum_{i=1}^{N} \lambda_i Q^{(i)} Q^{(i)} + \cdots\right] \boldsymbol{b} \tag{17a}$$

and

$$E[\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^T] = \left[B + \sum_{i=1}^{N} \lambda_i \left(Q^{(i)} Q^{(i)} B + Q^{(i)} B Q^{(i)T} + B Q^{(i)T} Q^{(i)T}\right) + \cdots\right] \tag{17b}$$

The diagonal terms of Eq. (17b) give $E[u_i^2]$, where u_i is the ith component of vector $\bar{u}$. The standard deviation of u_i is given as

$$\sigma_{u_i} = \sqrt{[E[u_i^2] - (E[u_i])^2]}$$

The formulation presented is quite general, and the various matrices presented can be computed in arbitrary bases. In the next section, wavelet bases are used to generate these matrices.

III. Wavelet Stochastic FEM

The wavelet-based stochastic FEM presented here is similar to other transform methods in that the functions are approximated into a basis and the computations are performed in the new system of coordinates. In fact, wavelets can be replaced from the present formulation by any other set of orthogonal functions without major modifications. However, wavelets are used here because they lead to formulations that are computationally efficient. In this study, the equations are formulated in terms of extended Haar wavelet (EHW) bases defined below. Other wavelet bases such as wavelet-packet bases, local trigonometric bases, Daubechies' wavelets, and wavelets based on the interval[2,8] also can be used for the task.

To develop the model, we begin with the Haar function, which is defined as

$$\psi(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

A family of orthogonal Haar wavelets $\psi_{m,n}(x)$ is obtained using Eq. (18) and the following equation:

$$\psi_{m,n}(x) = A_m \psi(2^m x - n), \qquad m, n \in Z \tag{19}$$

where m and n are the dilation and the translational numbers, and A_m is the amplitude of the function. When $A_m = 2^{m/2}$, we get a family of orthonormal Haar wavelets, and when $A_m = 1$, we get Haar wavelets of unit amplitude. Several investigators[1,2] have considered $A_m = 2^{m/2}$. However, use of $A_m = 1$ leads to a more computationally efficient formulation than use of $A_m = 2^{m/2}$. This is because $A_m = 1$ eliminates the need to carry the coefficient and the amplitude terms separately. For this reason, we take $A_m = 1$.

Wavelets $\psi_{m,n}(x)$ $(m, n = 0, 1, \ldots,)$ form a basis for square integrable functions defined over a real line. This is not the case if the function and the wavelets are defined over a bounded domain. To overcome this difficulty, a set of EHW bases $\psi_i(x)$ $(i = 0, 1, \ldots,)$ is defined as

$$\psi_0(x) = 1 \tag{20}$$

$$\psi_i(x) = \psi_{m,n}(x) \tag{20}$$

$$i = 2^m + n, \qquad m = 0, 1, \ldots, \qquad n = 0, \ldots, 2^m - 1 \tag{21}$$

It can be shown that this set forms a complete set of extended orthogonal wavelets over the domain [0, 1], and the orthogonal relationship is defined as

$$\int_0^1 \psi_i(x)\psi_j(x)\,dx = d_{(i)}\delta_{ij} \tag{22}$$

where, for orthonormal wavelets, $d_{(i)} = 1$, and for orthogonal wavelets it is an appropriate constant. For approximate solution, the eigenfunction $\phi_i(x)$ and the correlation function $R_f(x, u)$ in Eq. (4) are represented in terms of EHW bases as

$$\phi_i(x) = \sum_{j=0}^{n} z_{ji}\psi_j(x) = \Psi^T(x)z_i \tag{23}$$

$$R_f(x, u) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\psi_i(x)\psi_j(u)\,du = \Psi^T(x)A\Psi(u) \tag{24}$$

where $n + 1$ is the number of EHW bases selected,

$$\Psi(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_n(x)]^T \tag{25}$$

$$z_i = [z_{0i}, z_{1i}, \ldots, z_{ni}]^T \tag{26}$$

A is an $(n+1) \times (n+1)$ matrix of coefficients a_{ij}, which are given as

$$a_{ij} = \frac{1}{d_{(i)}d_{(j)}} \int_0^1 \int_0^1 R_f(x, u)\psi_i(x)\psi_j(u)\,dx\,du \tag{27}$$

and the superscript T represents the transpose of a vector (or a matrix). The double integral in Eq. (27) represents two-dimensional wavelet transform of the function $R_f(x, u)$.

Substituting Eqs. (23) and (24) into Eq. (4) and using the orthogonality condition given by Eq. (22), we obtain

$$\lambda_i \Psi^T(x)z_i = \Psi^T(x)ADz_i \tag{28}$$

where

$$D = \begin{bmatrix} d_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}^T \tag{29}$$

is a diagonal matrix. Substituting $z_i' = D^{1/2}z_i$ into Eq. (28) and equating the coefficients of $\Psi(x)$, we obtain

$$\lambda_i z_i' = A' z_i' \tag{30}$$

where $A' = D^{1/2}AD^{1/2}$. Equation (30) is the finite dimensional representation of the integral eigenvalue problem given by Eq. (4). Note that subscript i in Eq. (30) can take any value between 0 and n (including 0 and n), and therefore a complete solution of Eq. (30) provides $n + 1$ eigenvalues and corresponding $n + 1$ eigenvectors. Once the eigenvalues λ_i and the eigenvectors z_i' $(i = 0, 1, \ldots, n)$ have been computed, the eigenfunctions $\phi_i(x)$ can be computed from the following equation:

$$\phi_i(x) = \Psi^T(x)D^{-\frac{1}{2}}z_i' \tag{31}$$

Observe that the coefficients a_{ij} [Eq. (27)] and the function values $\phi_i(x)$ [Eq. (31)] can be computed efficiently in wavelet bases.[1,8] Equation (31) considers only one random process. If there are other random processes, then the same scheme can be applied to each process.

IV. Computational Strategy and Operation Counts

To demonstrate that the present scheme provides an efficient algorithm to generate the global matrices $K^{(i)}$ without requiring additional subroutines specific to the matrices, consider Eq. (9) once again. As stated in Sec. II, this equation is only a symbolic representation of matrix $K^{(i)}$, and it does not show how the matrix is generated. In practice, element matrices are generated and assembled to obtain the global matrices. To obtain an efficient algorithm to generate the global matrices $K^{(i)}$, divide the beam structure into $n = 2^m$ number of equal elements and write the equation for the kth stochastic-element matrix associated with eigenfunction i as

$$K_k^{(i)} = \int_{(k-1)/n}^{k/n} B_e^T E I_{(0)} \phi_i(x) B_e\,dx, \qquad k = 1, \ldots, N \tag{32}$$

where B_e represents matrix B for the element. Note that matrix $K_k^{(i)}$ generally is not available in the existing deterministic finite element code, and subroutines must be developed to compute this matrix.

In the discussion to follow, we assume that the beam element consists of two nodes and two degrees of freedom at each node. Our derivation shows that, when analytical expressions for $\phi_i(x)$ are used, the computation of $K_k^{(i)}$ for the element considered requires approximately 40 additions and subtractions and 80 multiplications and divisions. Thus, the semianalytical scheme requires $50\,nN$ additions and subtractions and $80\,nN$ multiplications and divisions to generate the matrices $K^{(i)}$, where N is the number of K–L terms. Note that 10 extra additions and subtractions are needed for each element during the assembly of the element matrices.

For EHW, $\phi_i(x) = \phi_{k,i}$ $(i = 1, \ldots, n)$ are constants over each element. Therefore, it follows that

$$K_k^{(i)} = \phi_{k,i} \int_{(k-1)/n}^{k/n} B_e^T E I_{(0)} B_e\,dx = \phi_{k,i} K_{\text{element}}^{(0)} \tag{33}$$

where $K^{(0)}_{\text{element}}$ is the deterministic element stiffness matrix that is readily available in most finite element codes, and $\phi_{k,i}$ is the value of the ith eigenfunction over the element k. Thus, matrix $K^{(i)}_k$ can be generated by multiplying $K^{(0)}_{\text{element}}$ by $\phi_{k,i}$. This requires 10 multiplications and divisions and no additions and subtractions. This indicates that the wavelet scheme requires $10nN$ additions and subtractions and $10nN$ multiplications and divisions to generate the matrices $K^{(i)}$.

Note that the assembly process for stochastic matrices is the same as that for deterministic matrices. Therefore, the generation of stochastic matrices $K^{(i)}$ $(i = 1, \ldots, N)$ requires only a small modification of the existing codes. The boundary conditions now can be imposed to obtain $\bar{K}^{(i)}$. Once $\bar{K}^{(i)}$, $i = 1, \ldots, N$, are known, Eqs. (12) and (17) can be used to compute $\bar{u}$ in terms of ϵ_i. In practice, $Q_{(i)}$, $i = 1, \ldots, N$, are never computed. Instead, observe that Eq. (17) contains terms like $Q^{(i)}\boldsymbol{b}$. For simplicity, we define $Q^{(i)}\boldsymbol{b} = \boldsymbol{b}_i$, which, after substitution of the value of $Q^{(i)}$ from Eq. (12) and manipulation of the resulting equation, leads to

$$\bar{K}^{(0)}\boldsymbol{b}_i = \bar{K}^{(i)}\boldsymbol{b} \tag{34}$$

Vector $\bar{K}^{(i)}\boldsymbol{b}$ can be generated in two ways: first, by direct multiplication of $\bar{K}^{(i)}$ and $\boldsymbol{b}$; and second, by multiplying $\bar{K}^{(i)}$ and $\boldsymbol{b}$ at element level and then assembling the resulting vector. For the semianalytical method, the first approach turns out to be advantageous. Because $\bar{K}^{(i)}$ is banded, computation of $\bar{K}^{(i)}\boldsymbol{b}$ $(i = 1, \ldots, N)$ by direct multiplication requires $(14n + 4)N$ additions and subtractions and $(14n + 4)N$ multiplications and divisions. This is also true for the wavelet scheme if the first approach is used. On the other hand, in the wavelet scheme, the second approach requires $(12n + 4nN)$ additions and subtractions and $(16n + 4nN)$ multiplications and divisions. Observe that the wavelet scheme requires fewer computations for several terms.

Equation (34) can be solved for $\boldsymbol{b}_i$ using the LU factorization technique. Because the matrix $\bar{K}^{(0)}$ remains the same for all right-hand-side vectors of type $\bar{K}^{(i)}\boldsymbol{b}$, the LU factorization of $\bar{K}^{(0)}$ needs to be generated only once, and the same factorization can be used over and over to compute various terms in Eq. (17), and computation of $[\bar{K}^{(0)}]^{-1}$ is not necessary. The first- and the second-moment characteristics of the displacement vector $\boldsymbol{u}$ can be obtained by computing $E[\bar{\boldsymbol{u}}]$ and $E[u_i^2]$ terms and imposing the condition given by Eq. (5).

V. Numerical Examples

To demonstrate the feasibility and accuracy of the formulation, two beam structures, a cantilever with a point load and a simply supported beam with distributed load, are considered. It is assumed that $EI(x)$ varies randomly and is given by Eq. (2). The characteristics of the random function $f(x)$ in Eq. (2) are defined by zero mean function and the following triangular correlation function:

$$R_f(x, u) = \begin{cases} \sigma^2[1 - (1/b)|x - u|] & \text{if} \quad |x - u| \le b \\ 0 & \text{if} \quad |x - u| > b \end{cases} \tag{35}$$

where σ is the variance of the process and b is the correlation length. Equations are scaled so that limits of the integrals range from 0 to 1. For numerical purposes, the following values are used: $PL^3/(EI)_0 = 0.03$, $qL^3/(EI)_0 = 1.0$, $\sigma = 0.1$, and $b = 1$. The beam is divided into 16 elements and the integral eigenvalue problem is solved using 16 wavelets.

Each problem is solved using the Monte Carlo scheme, the semianalytical scheme,[22] and the wavelet-based scheme. For simplicity in the discussion, the following abbreviations are used: AM (analytical Monte Carlo) and AN (analytical Neumann) for Monte Carlo simulation and Neumann expansion-based schemes, respectively, if matrices $K^{(i)}$ in Eq. (7) are generated using the analytical expressions for $\phi_i(x)$; and WM (wavelet Monte Carlo) and WN (wavelet Neumann), respectively, for Monte Carlo simulation and Neumann expansion-based schemes if matrices $K^{(i)}$ in Eq. (7) are generated using the wavelets as discussed in the preceding sections. Numerical experiments are conducted with various numbers of terms in K–L expansion and Neumann series, and various sample sizes for Monte Carlo simulations. For each experiment, results are obtained for mean value and standard deviation of deflections. Some of the results of these simulations are presented here. In each Monte Carlo simulation presented, 10,000 sample sizes are considered.

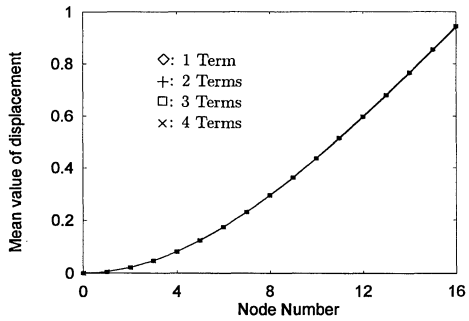

Fig. 1 Comparison of mean value of displacement for one-, two-, three-, and four-term K–L expansion WM scheme for cantilever beam problem.

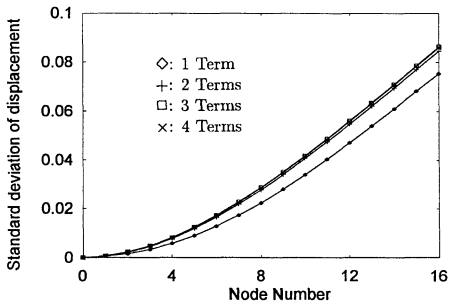

Fig. 2 Comparison of standard deviation of displacement for one-, two-, three-, and four-term K–L expansion WM scheme for cantilever beam problem.

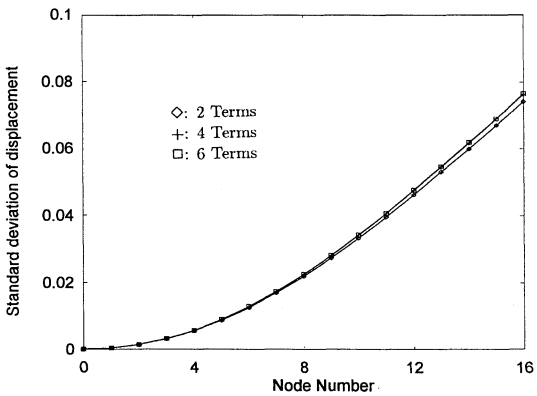

Fig. 3 Comparison of standard deviation of displacement for WN scheme with one-term K–L and two-, four-, and six-term Neumann expansion for cantilever beam problem.

The first set of results is for the cantilever beam subjected to a point load. Figures 1 and 2 show the results for the mean value and the standard deviation of deflections obtained for one-, two-, three-, and four-term K–L expansion WM scheme. Results for standard deviation of deflections using WN scheme for one-term K–L expansion and two-, four-, and six-term Neumann series are shown in Fig. 3. Results for standard deviation of deflections using WN scheme for one-, two-, and three-term K–L expansion and two-term Neumann series are shown in Fig. 4. The mean values and the standard deviations of deflections obtained using AM, AN, WM, and WN methods are compared in Figs. 5 and 6, respectively. In each method, three-term K–L expansions were used. Only a few results are presented for mean value of deflections because all mean value results are very close. It can be seen that results converge very rapidly as the number of terms in K–L expansion and Neumann series is increased. Furthermore, the results of this scheme compare well with those of Ghanem and Spanos.[22] Corresponding results for the simply supported beam with a distributed load are shown in Figs. 7–12. Results for this structure follow the same pattern as that of the cantilever beam with a point load.

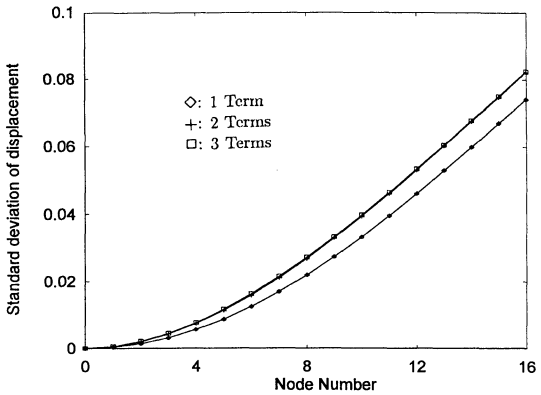

Fig. 4 Comparison of standard deviation of displacement for WN scheme with two-term Neumann and one-, two-, and three-term K–L expansion for cantilever beam problem.

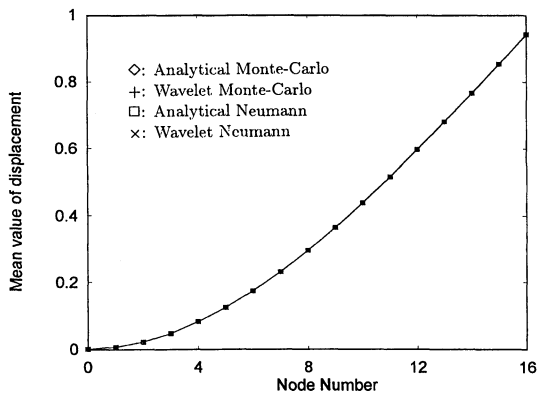

Fig. 5 Comparison of mean value of displacement obtained using AM, WM, AN, and WN methods for cantilever beam problem.

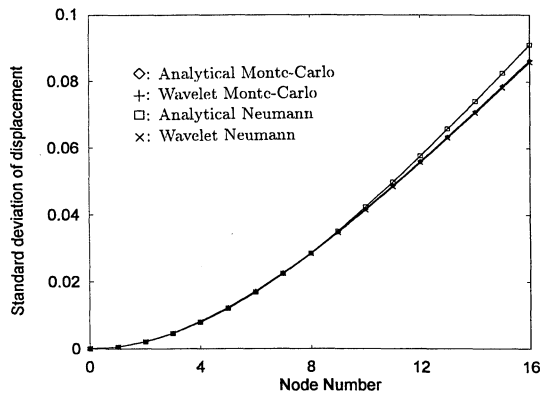

Fig. 6 Comparison of standard deviation of displacement obtained using AM, WM, AN, and WN methods for cantilever beam problem.

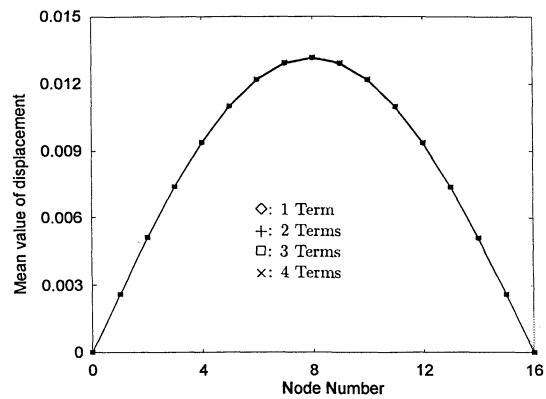

Fig. 7 Comparison of mean value of displacement for one-, two-, three-, and four-term K–L expansion WM scheme for simply supported beam problem.

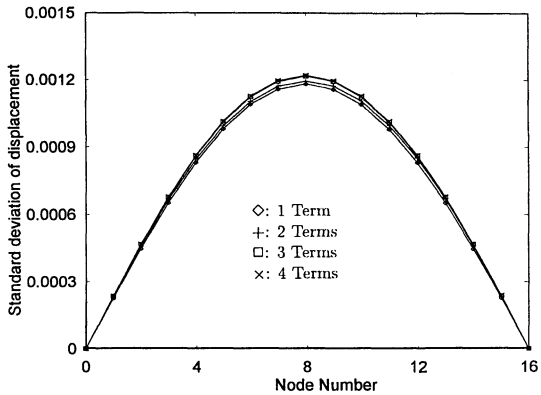

Fig. 8 Comparison of standard deviation of displacement for one-, two-, three-, and four-term K–L expansion WM scheme for simply supported beam problem.

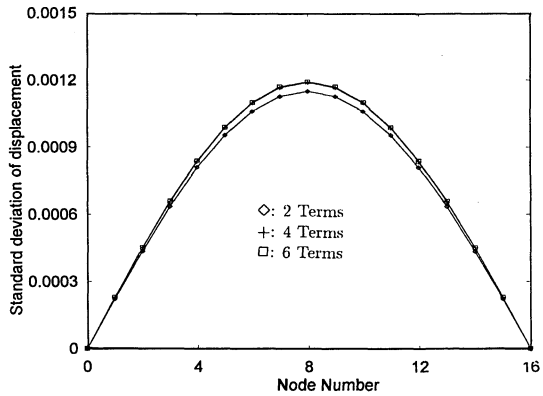

Fig. 9 Comparison of standard deviation of displacement for WN scheme with one-term K–L and two-, four-, and six-term Neumann expansion for simply supported beam problem.

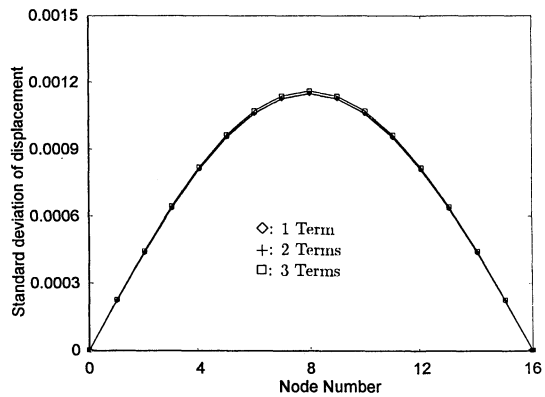

Fig. 10 Comparison of standard deviation of displacement for WN scheme with two-term Neumann and one-, two-, and three-term K–L expansion for simply supported beam problem.

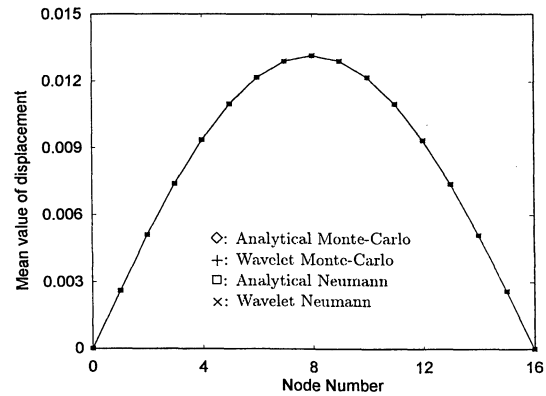

Fig. 11 Comparison of mean value of displacement obtained using AM, WM, AN, and WN methods for simply supported beam problem.

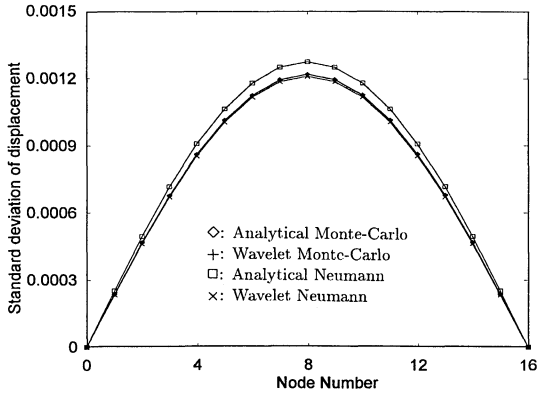

Fig. 12 Comparison of standard deviation of displacement obtained using AM, WM, AN, and WN methods for simply supported beam problem.

VI. Conclusions

A wavelet-based stochastic FEM for beam structures is presented. In this model, the random process representing the stochastic properties of the system is expressed by a mean value and a correlation function, and K–L expansion is used to approximate the process as a linear combination of eigenfunctions with uncorrelated random coefficients. It is shown that the integral eigenvalue problem can be solved, and stochastic finite element matrices can be generated efficiently using wavelet bases. The Neumann-series expansion helped in expressing the displacements in terms of random coefficients. Expectation operators are applied to appropriate functions to compute the mean and the standard deviation of displacements. The scheme is used to solve two beam structures, and the results are compared with the existing schemes. The present study shows that the results obtained using this scheme agree well with those of existing schemes.

Acknowledgments

This research was partially supported by the Dean's Office, College of Engineering, Southern Illinois University, and by NASA Lewis Research Center Grant 48503.

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